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Some Newton's Type Inequalities for Geometrically Relative Convex Functions

[1]Muhammad Aslam Noor, [1]Khalida Inayat Noor and [1*]Muhammad Uzair Awan

[1]Department of Mathematics,
COMSATS Institute of Information Technology,Pakistan

E-mail: awan.uzair@gmail.com

*Corresponding author

ABSTRACT

In this paper, we derive some Newton's type of integral inequalities for geometrically relative convex functions. Some special cases are also discussed.

Keywords: Convex functions, geometrically relative convex functions, Newton's inequalities.

1. Introduction

Let $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable function on I° where I° is the interior of I, and for the fourth derivative to be bounded on I°, that is, $\| f^{(4)} \|_\infty = \sup_{x\in(a,b)} | f^{(4)}(x) | < \infty$. Then the following inequality holds:

$$\left| \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{1}{6480} \| f^{(4)} \|_\infty (b-a)^4.$$

In recent years, theory of convex sets and convex functions has been extended and generalized using novel and innovative techniques. Youness

(1999) introduced and studied the relative convex sets and relative convex functions with respect to an arbitrary function. These relative convex sets and relative convex functions are nonconvex. It has been shown that these nonconvex functions have some nice properties, analogue to the convex functions. Noor (1988) proved that the optimality conditions of differentiable relative convex functions on the relative convex sets can be characterized by a class of variational inqualities. This class of variational inequalities is called generalize variational inequalities introduced and studied by Noor (1988). For the formulation, applications, numerical methods and various other aspects of general variational inequalities, see Noor (1988), Noor (1988a) and the references therein. Noor, Noor and Awan (2013), Noor, Noor and Awan (2014) and Noor, Noor and Awan (2014a) have derived several Hermite-Hadamard type inequalities for different classes of relative convex functions. For some recent investigations on this subject, (see Cristescu and Lupsa. (2002), Dragomir and Pearce (2000), Dragomir, Pecaric and Persson (1995), Dragomir, Agarwal and Cerone (2000), Shuang, Yin and Qi (2013), Zhang, Ji and Qi (2012) and Zhang, Ji and Qi (2013)).

Motivated and inspired by onoing research in this field, we obtain some Newton type inequalities for geometrically relative convex functions, which is the main motivation of this paper. Several specail cases are also discussed. The interested readers are encouraged to find applications of the relative convex sets and relative convex functions in various fields of pure and applied sciences.

2. Preliminaries

In this section, we recall some basic previously known concepts.

Definition 2.1. (Noor, Noor and Awan (2014)). *Let* $\mathcal{G} \subseteq (0,\infty)$. *Then* $\mathcal{G}$ *is said to be geometrically relative convex set, if there exists an arbitrary function* $g:\mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$(g(x))^t (g(y))^{1-t} \in \mathcal{G}, \quad \forall g(x), g(y) \in \mathcal{G}, t \in [0,1].$$

From this, it follows that

$$(g(x))^t (g(y))^{1-t} \le tg(x) + (1-t)g(y), \quad \forall g(x), g(y) \in \mathcal{G}, t \in [0,1].$$

Definition 2.2: (Noor, Noor and Awan (2014)). *A function* $f:\mathcal{G}\to\mathbb{R}$ *(on subintervals of* $(0,\infty)$*) is said to be geometrically relative convex function (* GG *-relative convex function) if there exists an arbitrary function* $g:\mathbb{R}^n\to\mathbb{R}^n$ *such that,*

$$f((g(x))^t(g(y))^{1-t})\le(f(g(x)))^t(f(g(y)))^{1-t}, \tag{1}$$

for all $g(x),g(y)\in\mathcal{G},t\in[0,1]$.

From (1), it follows that

$$\log f((g(x))^t(g(y))^{1-t})\le t\log f(g(x))+(1-t)\log f(g(y)),$$
$$\forall g(x),g(y)\in\mathcal{G},t\in[0,1]$$

Also

$$\begin{aligned}f((g(x))^t(g(y))^{1-t})&\le(f(g(x)))^t(f(g(y)))^{1-t}\\&\le tf(g(x))+(1-t)f(g(y)).\end{aligned}$$

This implies that every geometrically relative convex function (GG -relative convex function) is also GA-relative convex function, but the converse is not true.

For $t=\dfrac{1}{2}$ in (1), we have Jensen type of geometrically relative convex functions. That is

$$f\left(\sqrt{g(x)g(y)}\right)\le\sqrt{f(g(x))f(g(y))}.$$

Definition 2.3. (Noor, Noor and Awan (2014)). *A function* $f:\mathcal{G}\to\mathbb{R}$ *(on subintervals of* $(0,\infty)$*) is said to be* GA*-relative convex function if there exists an arbitrary function* $g:\mathbb{R}^n\to\mathbb{R}^n$ *such that,*

$$f((g(x))^t(g(y))^{1-t})\le tf(g(x))+(1-t)f(g(y)),$$

for all $g(x),g(y)\in\mathcal{G},t\in[0,1]$.

Definition 2.4. (Noor, Noor and Awan (2014)). *A function* $f:\mathcal{G}\to\mathbb{R}$ *(on subintervals of* $(0,\infty)$ *) is said to be logarithmic relative (* AG *-relative) convex function if there exists an arbitrary function* $g:\mathbb{R}^n\to\mathbb{R}^n$ *such that,*

$$f(t(g(x))+(1-t)(g(y)))\le (f(g(x)))^{t}(f(g(y)))^{(1-t)},$$

for all $g(x),g(y)\in\mathcal{G},t\in[0,1]$.

3. Main Results

In this section, we prove our main results. First of all we prove following result, which plays a key role in proving our main results. This result can be proved using the technique of Gao and Shi (2012). However for the sake of completeness and to convey the idea, we include the details of the proof. Throughout this section, we denote $I=\left[g(a),g(b)\right]$ is the interval and I^0 is the interior of I .

Let us denote

$$\begin{aligned}&K\left(f;g;a,b\right)\\&=\frac{1}{8}\left[f(g(a))+3f\left(\frac{2g(a)+g(b)}{3}\right)+3f\left(\frac{g(a)+2g(b)}{3}\right)+f(g(b))\right]\\&\quad-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}f(g(x))\mathrm{d}g(x)\end{aligned}$$

Lemma 3.1. *Let* $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ *be twice differentiable function on* I° *. Let* $f''\in[g(a),g(b)]$, *where* $g(a),g(b)\in I$ *with* $g(a)<g(b)$ *and* $g:\mathbb{R}^n\to\mathbb{R}^n$ *be an arbitrary function. Then*

$$\left|K\left(f;g;a,b\right)\right|=(g(b)-g(a))^2\int_0^1\mu(t)f''((1-t)g(a)+tg(b))\mathrm{d}t,$$

where

$$\mu(t)=\begin{cases} -\dfrac{t^2}{2}+\dfrac{t}{8}, & t\in\left[0,\dfrac{1}{3}\right) \\ -\dfrac{t^2}{2}+\dfrac{t}{2}-\dfrac{1}{8}, & t\in\left[\dfrac{1}{3},\dfrac{2}{3}\right). \\ -\dfrac{t^2}{2}+\dfrac{7t}{8}-\dfrac{3}{8}, & t\in\left[\dfrac{2}{3},1\right] \end{cases}$$

Proof. Let

$$\begin{aligned}\mathcal{I}&=\int_0^1\mu(t)f''((1-t)g(a)+tg(b))\mathrm{d}t\\
&=\int_0^{\frac{1}{3}}\left(-\frac{t^2}{2}+\frac{t}{8}\right)f''((1-t)g(a)+tg(b))\mathrm{d}t\\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(-\frac{t^2}{2}+\frac{t}{2}-\frac{t}{8}\right)f''((1-t)g(a)+tg(b))\mathrm{d}t\\
&+\int_{\frac{2}{3}}^{1}\left(-\frac{t^2}{2}+\frac{7t}{2}-\frac{3}{8}\right)f''((1-t)g(a)+tg(b))\mathrm{d}t\\
&=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned}\tag{4}$$

Now

$$\begin{aligned}\mathcal{I}_1&=\int_0^{\frac{1}{3}}\left(-\frac{t^2}{2}+\frac{t}{8}\right)f''((1-t)g(a)+tg(b))\mathrm{d}t\\
&=-\frac{1}{72(g(b)-g(a))}f'\left(\frac{2g(a)+g(b)}{3}\right)\\
&+\frac{5}{24(g(b)-g(a))^2}f\left(\frac{2g(a)+g(b)}{3}\right)+\frac{1}{8(g(b)-g(a))^2}f(g(a))\\
&-\frac{1}{(g(b)-g(a))^2}\int_0^{\frac{1}{3}}f((1-t)g(a)+tg(b))\mathrm{d}t,\end{aligned}\tag{5}$$

and

$$\begin{aligned}\mathcal{I}_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left(-\frac{t^2}{2}+\frac{t}{2}-\frac{1}{8}\right) f''((1-t)g(a)+tg(b))dt \\ &= \frac{1}{72(g(b)-g(a))} f'\left(\frac{2g(a)+g(b)}{3}\right) \\ &\quad -\frac{1}{72(g(b)-g(a))} f'\left(\frac{g(a)+2g(b)}{3}\right) \\ &\quad +\frac{1}{6(g(b)-g(a))^2}\left[f\left(\frac{2g(a)+g(b)}{3}\right)+f\left(\frac{g(a)+2g(b)}{3}\right)\right] \\ &\quad -\frac{1}{(g(b)-g(a))^2}\int_{\frac{1}{3}}^{\frac{2}{3}} f((1-t)g(a)+tg(b))\mathrm{d}t. \qquad (6)\end{aligned}$$

Similarly

$$\begin{aligned}\mathcal{I}_3 &= \int_{\frac{2}{3}}^{1} \left(-\frac{t^2}{2}+\frac{7t}{8}-\frac{3}{8}\right) f''((1-t)g(a)+tg(b))\mathrm{d}t \\ &= \frac{1}{72(g(b)-g(a))} f'\left(\frac{g(a)+2g(b)}{3}\right) \\ &\quad +\frac{5}{24(g(b)-g(a))^2} f\left(\frac{g(a)+2g(b)}{3}\right) \\ &\quad +\frac{1}{8(g(b)-g(a))^2} f(g(b)) \\ &\quad -\frac{1}{(g(b)-g(a))^2}\int_{\frac{2}{3}}^{1} f((1-t)g(a)+tg(b))\mathrm{d}t. \qquad (7)\end{aligned}$$

Using (5), (6) and (7) in (4), and then multiplying both sides $(g(b)-g(a))^2$ completes the proof. □

Now using Lemma 3.1, we prove some Newton's type integral inequalities.

Theorem 3.2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be twice differentiable function on* I°. *Let* $f'' \in [g(a), g(b)]$, *where* $g(a), g(b) \in I$ *with* $g(a) < g(b)$ *and* $g : \mathbb{R}^n \to \mathbb{R}^n$ *be an arbitrary function. If* $|f''|$ *is decreasing and geometrically relative convex function. Then*

$$\left|K\left(f; g; a,b\right)\right| \le (g(b)-g(a))^2 \, |f''(g(a))| \, \mathfrak{F}(t),$$

where $w = \dfrac{|f''(g(b))|^t}{|f''(g(a))|^t}$ and $\mathfrak{F}(t) = \int_0^1 |\mu(t)| \, w dt.$

Proof. Using Lemma 3.1 and the fact that $|f''|$ is decreasing and geometrically relative convex function, we have

$$\begin{aligned}
&\left|K\left(f; g; a,b\right)\right| \\
&\le (g(b)-g(a))^2 \int_0^1 |\mu(t)| \, |f''((1-t)g(a)+tg(b))| \, \mathrm{d}t \\
&\le (g(b)-g(a))^2 \int_0^1 |\mu(t)| \, |f''((g(a))^{(1-t)}(g(b))^t)| \, \mathrm{d}t \\
&\le (g(b)-g(a))^2 \int_0^1 |\mu(t)| \, |(f''(g(a)))^{(1-t)}(f''(g(b)))^t| \, \mathrm{d}t \\
&= (g(b)-g(a))^2 \, |f''(g(a))| \int_0^1 |\mu(t)| \frac{|f''(g(b))|^t}{|f''(g(a))|^t} \mathrm{d}t \\
&= (g(b)-g(a))^2 \, |f''(g(a))| \, \mathfrak{F}(t).
\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be twice differentiable function on* I°. *Let* $f'' \in [g(a), g(b)]$, *where* $g(a), g(b) \in I$ *with* $g(a) < g(b)$ *and* $g : \mathbb{R}^n \to \mathbb{R}^n$ *be an arbitrary function. If* $|f''|^q$ *is decreasing and geometrically relative convex function, where* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\left|K\left(f;g;a,b\right)\right|$$
$$\leq (g(b)-g(a))^2\,|\,f''(g(a))|\left(\int_0^1 |\,\mu(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 w^{qt}dt\right)^{\frac{1}{q}},$$

where $w=\dfrac{|\,f''(g(b))|^t}{|\,f''(g(a))|^t}$.

Proof. Using Lemma 3.1, well known Holder's inequality and the fact that $|\,f''|^q$ is decreasing and geometrically relative convex function, we have

$$\left|K\left(f;g;a,b\right)\right|$$
$$\leq (g(b)-g(a))^2\left(\int_0^1 |\,\mu(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |\,f''((1-t)g(a)+tg(b))|^q\,dt\right)^{\frac{1}{q}}$$
$$\leq (g(b)-g(a))^2\left(\int_0^1 |\,\mu(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |\,f''((g(a))^{(1-t)}(g(b))^t)|^q\,dt\right)^{\frac{1}{q}}$$
$$\leq (g(b)-g(a))^2\left(\int_0^1 |\,\mu(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |(\,f''((g(a)))^{(1-t)}(\,f''(g(b)))^t)|^q\,dt\right)^{\frac{1}{q}}$$
$$\leq (g(b)-g(a))^2\,|\,f''(g(a))|\left(\int_0^1 |\,\mu(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 w^{qt}dt\right)^{\frac{1}{q}},$$

where $\int_0^1 |\,\mu(t)|^p\,dt$ may be calculated using maple. This completes the proof.

Theorem 3.4. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I°. Let $f''\in[g(a),g(b)]$, where $g(a),g(b)\in I$ with $g(a)<g(b)$ and $g:\mathbb{R}^n\to\mathbb{R}^n$ be an arbitrary function. If $|\,f''|^q$ is decreasing and geometrically relative convex function, where $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|K\left(f;g;a,b\right)\right|\leq (g(b)-g(a))^2\,|\,f''(g(a))|\left(\int_0^1 |(\mu(t)\,|\,w^t)^q\,dt\right)^{\frac{1}{q}},$$

where $w=\frac{|f''(g(b))|^t}{|f''(g(a))|^t}$.

Proof. Using Lemma 3.1, well known Holder's inequality and the fact that $|f''|^q$ is decreasing and geometrically relative convex function, we have

$$\begin{aligned}
&\left|K\left(f;g;a,b\right)\right| \\
&\leq (g(b)-g(a))^2\left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\int_0^1 |\mu(t)|^q|\ f''((1-t)g(a)+tg(b))|^q\ dt\right)^{\frac{1}{q}} \\
&\leq (g(b)-g(a))^2\left(\int_0^1 |\mu(t)|^q|\ f''((g(a))^{(1-t)}(g(b))^t)|^q\ dt\right)^{\frac{1}{q}} \\
&\leq (g(b)-g(a))^2\left(\int_0^1 |\mu(t)|^q|(f''((g(a)))^{(1-t)}(f''(g(b)))^t)|^q\ dt\right)^{\frac{1}{q}} \\
&\leq (g(b)-g(a))^2\ |f''(g(a))|\left(\int_0^1 |(\mu(t)|w^t)^q dt\right)^{\frac{1}{q}},
\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let* $f: I\subset\mathbb{R}\to\mathbb{R}$ *be twice differentiable function on* I°. *Let* $f''\in[g(a),g(b)]$, *where* $g(a),g(b)\in I$ *with* $g(a)<g(b)$ *and* $g:\mathbb{R}^n\to\mathbb{R}^n$ *be an arbitrary function. If* $|f''|^q$ *is decreasing and geometrically relative convex function, where* $q>1$, *then*

$$\left|K\left(f;g;a,b\right)\right|\leq (g(b)-g(a))^2\ |f''(g(a))|\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left(\int_0^1 |\mu(t)|\,w^{qt}dt\right)^{\frac{1}{q}},$$

where $w=\frac{|f''(g(b))|^t}{|f''(g(a))|^t}$.

Proof. Using Lemma 3.1, well known Power mean inequality and the fact that $|f''|^q$ is decreasing and geometrically relative convex function, we have

$$
\begin{aligned}
&\left|K\left(f;g;a,b\right)\right| \\
&\le (g(b)-g(a))^2\left(\int_0^1 |\mu(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |\mu(t)||\,f''((1-t)g(a)+tg(b))|^q\,dt\right)^{\frac{1}{q}} \\
&\le (g(b)-g(a))^2\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left(\int_0^1 |\mu(t)||\,f''((g(a))^{(1-t)}(g(b))^t)|^q\,dt\right)^{\frac{1}{q}} \\
&\le (g(b)-g(a))^2\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left(\int_0^1 |\mu(t)||\,(f''((g(a)))^{(1-t)}(f''(g(b)))^t)|^q\,dt\right)^{\frac{1}{q}} \\
&\le (g(b)-g(a))^2\,|\,f''(g(a))|\left(\frac{1}{192}\right)^{1-\frac{1}{q}}\left(\int_0^1 |\mu(t)|\,w^{qt}dt\right)^{\frac{1}{q}},
\end{aligned}
$$

This completes the proof. □

Remark 3.6. *For* $q=1$*, Theorem 3.5 reduces to Theorem 3.2.*
It is worth mentioning here that for $g=I$*, where* I *is identity function, our results reduce to the results for geometrically relative convex function in the classical sense. To the best of our knowledge, these results seem to be new ones.*

4. Conclusion

We have considered and investigated the class of geometrically relative convex functions. We have proved an auxiliary result for the differentiable functions. Using this result, we have derived several new Newton's type inequalities for geometrically relative convex functions. It is shown that results obtained in this paper represent refinement and improvement of several new and known results, which can be obtained as special cases.

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